The Solitaire Memory Game

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Abstract

Memory is a popular card game played by people of all ages around the world. Given a set of n pairs of cards laid out face down, a player may in one move turn over two cards one after another. If the cards form a pair, they get collected off the table, else they get turned over again. For the 2–player game, where players alternate in moves and win if they collect more pairs, an optimal strategy is known and well-studied. Here we consider the 1–player *solitaire* game, where the goal is to need as few moves as possible to collect all cards off the table. We prove that an optimal strategy needs less than $1.75 \cdot n$ moves in expectation. Furthermore we prove the lower bound that every strategy needs at least $1.5 \cdot n - 1$ moves for arbitrary n in expectation. Intensive numerical calculations lead to the new interesting conjecture that an optimal strategy has a competitive ratio of $1.613603 < c < 1.613706$. In particular, we study games where already k different cards are known to the player. We prove that an optimal strategy needs at least $1.5n - 0.5k - 1$ and at most $2n - k$ moves in expectation to finish such a game. If one is interested in a strategy that guarantees to finish a solitaire Memory game, then $2n - 1$ moves are both necessary and sufficient.

1 Introduction

Memory (or cf. [6]: Concentration, Pairs, Pelmanism, Pexeso, Shinkei-suijaku) is a popular card game that requires good memorization skills. A deck of pairs of cards is shuffled and laid out face down on a table. The goal of the game is to collect all pairs of cards: In the solitaire version, in as few moves as possible, and with multiple players, to collect more pairs than each opponent. In a single move, a player may first turn over a single card and then another card – if both cards form a pair, they are collected, else they are turned over again.While children often have an advantage in the game due to their innate memory skills [5], professional tournament games with 32 or 31 pairs (cf. [7]) are often played by using strategies with a quite sophisticated mathematical background [9].

The 2-player version (where the goal is to collect more pairs than the opponent) is quite well studied and an optimal strategy for n pairs is known due to Zwick and Paterson [11]. It is however not known how many moves the solitaire version of the game needs for a deck of n cards with an optimal strategy.

1.1 Our Results

We prove that an optimal strategy for the solitaire version needs between $1.5 \cdot n - 1$ and $1.75 \cdot n$ expected moves to collect all n pairs. Intensive numerical calculations lead to the new interesting conjecture that for large n, the number of expected moves is between $1.613603 \cdot n$ and $1.613706 \cdot n$. For solitaire Memory games in the tournament version (with 32 or 31 pairs) the expected number of needed moves with an optimal strategy are $\approx 1.5977 \cdot 32$ or $\approx 1.5972 \cdot 31$. If the player already knows k different cards, then we show that an optimal strategy needs at least $1.5n - 0.5k - 1$ and at most $2n - k$ more moves in expectation. If one is interested in the worst-case number of moves to finish a new game, then $2n - 1$ moves are both necessary and sufficient.

1.2 Related Work

Combinatorial games and problems are a popular subject throughout history. While some problems like the *Seven Bridges of Königsberg* by Euler in 1735 started whole branches of modern mathematics, others like the *Icosian game* by Hamilton in 1857 are still million dollar prize problems in their general form in todays world[1]. It is also still an expanding subject in mathematics and computer science, since, according to Fraenkel [2], "*The combinatorial games community is growing in quantity and quality!*". As a proof, he lists 1700 publications (e.g., [10]) in his "*Selected Bibliography*". Many of the listed games can be investigated with dynamic programming, we refer to [8] for an overview.

One of the first scientific publications about Memory was by Kirkpatrick in 1954 [5], where he considered a variant with 26 pairs. He argued that one needs to remember around seven or eight cards in the beginning of the game and that the final phase of the game usually determines the winner of the participating players.

Zwick and Paterson considered the Memory game as a 2-player variant for any number of $n \in \mathbb{N}$ pairs in [11]. They searched for a strategy that maximizes the expected gain of pairs for a player. While their optimal strategy is easy to memorize, the proof of the optimality is "*extremely involved*" [3]. Zwick and Paterson also note that Gerez and Göbel had previously empirically found the optimal strategy in a report [4], where they also consider a variant where it is not allowed to collect two already known cards (that do not form a pair) in a single move. The optimal strategy for the 2-player Memory game can be summarized as follows [9]:

- Let $k \geq 0$ be the number of known cards currently on the table.
- Let $n \geq 1$ be the number of pairs currently on the table.
- If you can collect a known pair, do so. Else:
  - If $n + k$ is odd and $k \geq 2(n+1)/3$, turn over two cards that are already known.
  - If $n + k$ is even and $k \geq 1$ or if $k = 1$ and $n = 6$, turn over a new card. If you can now collect a pair, do so. Else turn over an already known card next.
  - Else turn over two new cards in the move.

As another variant, the 2-player Memory game was also considered in a modified version with finite memorization skills and an uncountable number of cards in [3].

2 Overview of the Paper

In Section 3 we describe the model for solitaire Memory, followed by an optimal strategy in Section 4. In Section 5 we show that in the worst case, $2n - 1$ moves are both necessary and sufficient. The main results of the paper are presented in Section 6 and 7: Section 6 discusses the expected number of moves needed for solitaire Memory. In Section 7 we present interesting new conjectures for large games based on extensive numerical investigations. Variations of the model are discussed in Section 8. The numerical calculations were done using *Mathematica*®9.

[1] See `http://www.claymath.org/millennium/P_vs_NP/`. Although one could argue that a proof would be worth much more than a mere million dollars.

3 The Model for Solitaire Memory

The game starts with $2n$ cards laid out face down on a table, so that a player can only see the identical back sides. W.l.o.g. each card is labeled with a natural number from 1 to n on its front side, with each label appearing exactly twice. Two cards with the same label form a pair. The $2n$ cards are shuffled uniformly at random beforehand, but the position of each uncollected card stays fixed during the game once it is laid out. We can therefore assume w.l.o.g. that it does not matter which of the yet not turned over cards will be turned over next, if the player desires to turn over a not yet turned over card. If a card has never been turned over yet, we call it unknown. If a card has been turned over, but not yet collected, we call it known.

The sole player makes a move as follows: first she turns over one card, then another card. If both cards have the same label, then the pair is collected, else both cards are put back into the same position face down. The goal of the game is to collect all pairs in as few moves as possible. We assume that the player has perfect memorization capabilities, meaning that she can remember all moves and the corresponding cards and their positions.

We assume that if the player has collected a pair in a move, then her next move is not free. We will later show that both versions of the game always differ by $n-1$ moves.

4 An Optimal Strategy for Solitaire Memory

When making a move, the player can turn over two cards after another. When turning over the first card in a move, the player can decide to turn over a card that is already known from a previous move or turn over a card that is yet unknown – the same is true for the second card in a move. We define a strategy for Solitaire Memory to be optimal, if it minimizes the number of moves needed. By case distinction we can easily deduce that the following naïve strategy is an optimal strategy:

Lemma 4.1. *The following strategy is an optimal strategy for solitaire Memory:*

While there are still cards left on the table, do the following in each move:

1. *If a matching pair of cards with the same label is known, turn them over and collect the pair.*
2. *Else turn over an unknown card:*
   (a) *If a card with the same label is known, turn over that card too and collect the pair.*
   (b) *Else turn over a second unknown card.*

Proof. When a pair of cards is known before a move, then it does not matter if it is collected in the next move or at some later point, since it will always take exactly one move to do so. In a similar fashion, it never makes sense to turn over two cards that are already known and not form a pair, since this will only increase the number of turns by one. We note that in the 2-player variant of Memory, this move is sometimes useful, see [11].

We can assume w.l.o.g. that for all strategies that finish the game, that the i-th unknown card that will be turned over is the same (with $1 \leq i \leq 2n$), since the cards were shuffled uniformly at random beforehand. If we do not know two cards with the same label (which we then can immediately collect, see the paragraph above), then we have $0 \leq k \leq n$ known cards with pairwise different labels:

When the player turns over a known card first and then an unknown card, she might as well first turn over the unknown card and then the known card, the outcome is the same after the move is finished. However, if she first turns over the unknown card, then she can also choose to do something different in the second part of the move. If the first turned over card does not match any known cards, then opening an already known card in the second part of the move will not reveal any new information, while opening a second unknown card will reveal one more card – and might possibly even lead to a pair. If the first turned over card has a known match, then one can either collect a pair or open a second new card. We can show that collecting a pair has no better alternative in this case by amortized analysis:

Three different cases can happen for each pair of cards with the same label:

1. Both cards are turned over in one move. Then each card of the pair gets turned over once (two turn-overs in total).
2. Both cards are already known from previous moves and then turned over. Then each card of the pair gets turned over twice (four turn-overs in total).
3. One card of the pair is previously known and the second card of the pair gets turned over in the first turn over of a new move. Then turning over the known partner of the card in this new move will result in one card getting turned over twice and one card getting turned over once – in average each of the two cards gets turned over 1.5–times.

At the end of a game, the number of moves is equivalent to the sum of the number of times each card got turned over divided by two – since in each move, we can turn over two cards. When the first turned over card in a move has an already known matching partner, then collecting this pair costs three turn-overs.

Might it be more efficient to not make the pair and instead open a second card? In this case, each card from the pair needs to be turned over twice, costing in total four turn-over for that pair (the most costly case of the three mentioned above). However, now we also know a new card – which could either have a known matching partner or not. If it has a known partner, then another move is needed to collect the pair, costing four turn-overs for this pair – again, the most costly possible case for this pair. Collecting both pairs has therefore a cost of 8 turn-overs in total. If the card has no known partner, then it could be that its partner is discovered in the first move of some step in the future, meaning that collecting this pair could cost only 3 turn-overs (less is not possible) – or combined with the first pair, 7 in total.

However, we will not need more than 4 turn-overs for the second pair (unless we play irrationally). So if we make the first pair in only 3 turn-overs, then we will not need more than 7 turn-overs for both pairs. This means there is no better strategy in this case than always collecting a pair if the first turned over card in a move has an already known matching card. □

5 Deterministic Memory – The Worst Case

In some sense Memory is a game of chance, since the player usually does not know beforehand if she can collect a pair of cards in the next move. But she can try to minimize the number of expected moves in total. However, what if we look at the worst case that can happen – how many moves are needed to always finish a game?

Lemma 5.1. *No strategy can guarantee to collect all pairs in a solitaire Memory game in less than $2n-1$ moves.*

Proof. We can assume that if no uncollected pairs are known to the player, then she will turn over a new card in the first step of her move. We can also assume that if the first card has no known matching partner, that then the player will turn over another unknown card as the second step of her move. Deviating from these rules would not decrease the number of needed moves. In a similar fashion, we can assume that if a pair of cards is known and not collected after a move, that the player will collect them at the end – doing it at an earlier point would incur the same costs of one move per pair.

The cards could be positioned in such a way, that the player will open the cards in the following order:

$$(1,2),(3,1),(4,2),(5,3),\ldots,$$

$$(n-2,n-4),(n-1,n-3),(n,n-2),(n,n-1)\ .$$

From the first $\{(1,2)\}$ to the $(n-1)$th move $\{(n,n-2)\}$, the first card that is turned over is always an unknown card that has no known matching partner. Therefore, the only viable option is to turn over a second card during that move. From the second to the $(n-1)$th move, this reveals a card that has a known matching partner from a previous move. Also in the first move, no pair is collected. Therefore, the player needs at least $(n-1)$ moves where no pair is made. Since collecting the pairs needs n additional moves at the end, each strategy needs at least $(n-1)+n=2n-1$ moves to finish collecting all pairs. □

Lemma 5.2. *There exists a strategy that can guarantee to collect all pairs in a solitaire Memory game in at most $2n-1$ moves.*

Proof. We use the following strategy: First the player turns over $(2n-2)$ cards in $(n-1)$ moves. Due to the pigeonhole principle, the player now knows the location of at least $(n-2)$ pairs (or has collected some of them already), since only two cards were not turned over yet. In the worst case, we can now collect these $(n-2)$ pairs with another $(n-2)$ moves, needing $(n-1)+(n-2)=2n-3$ moves in total so far. The player now turns over one of the last two remaining cards. If the players knows the location of the matching partner, she can collect this pair. Else, since only one card was not turned over yet, this last card is the matching partner. Since now only 2 cards are not collected on the table, they must be a pair. Adding these two moves to $2n-3$ previous moves results in total in $2n-1$ moves. □

Combining these lemmas gives the following Theorem:

Theorem 5.1. *For each optimal strategy and for each $n\in\mathbb{N}$, the worst case number of moves needed to finish the solitaire Memory game with n pairs is $2n-1$.*

6 The Expected Number of Moves for Solitaire Memory

Let us consider the optimal strategy for solitaire Memory from Chapter 4. If we know the location of an not yet collected pair, we would collect it in the next turn, but what happens when we do not know the location of a pair?

- Let n be the number of pairs left on the table.
- Let k be the number of known uncollected cards on the table.

Then the following probabilities hold for the first turn over of a move:

- The chance to turn over a known card is $\frac{k}{2n-k}$.
- The chance to turn over an unknown card is $\frac{2n-2k}{2n-k} = \frac{2(n-k)}{2n-k}$.

If we turn over a known card in the first part of a move, we collect a pair, but if we turn over an unknown card, then the following probabilities hold for the second part of a move:

- The chance to turn over a card that matches the first turned over card from this move is $\frac{1}{2n-k-1}$.
- The chance to turn over a card that matches one of the previously already known k cards is $\frac{k}{2n-k-1}$.
- The chance to turn over a card that matches no known cards (neither the first from this move nor the k previously known cards) is $\frac{2n-k-1-k-1}{2n-k-1} = \frac{2(n-k-1)}{2n-k-1}$.

Let us consider a small game of Memory with just three pairs for a starting example. The player could get lucky and finish in 3 moves. She could also be unlucky, and require $3 \cdot 2 - 1 = 5$ moves. If we apply the optimal strategy for solitaire Memory from Lemma 4.1, then a game with 6 cards can be modeled as the Markov chain seen in Figure 1. The expected number of needed moves is $13/3 \approx 1.444 \cdot 3$ for a game starting at (3,0,0). When starting it (2,0,0) it would be $8/3 \approx 1.333 \cdot 2$ and when starting at (1,0,0) it would be 1 move in expectation.

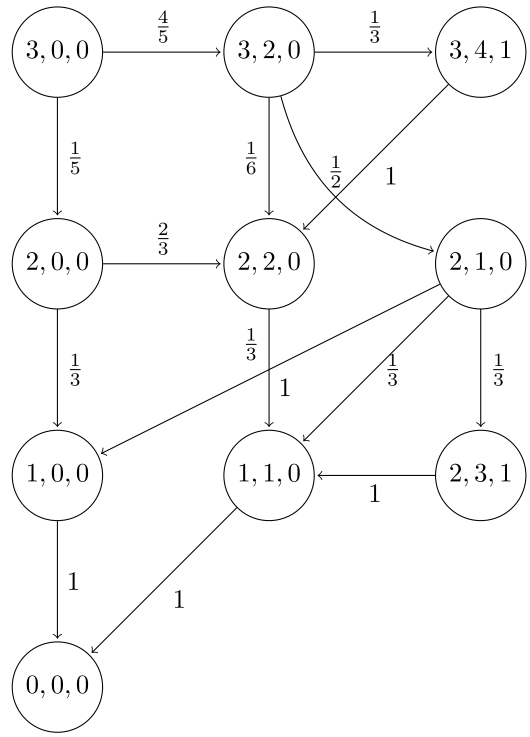

Figure 1: The triple in each node represents the number of pairs on the table, the number of known cards, the number of known uncollected pairs. The numbers on the edges represent the probability of the next move moving the game to the node at the end of the edge. The game starts at the top left node (3,0,0) and ends when all pairs are collected in the bottom left corner at the node (0,0,0).

Definition 6.1. *Let there be n pairs of cards on the table, where no pair is known. Let $0 \le k \le n$ be the number of known cards on the table. By $e_{n,k}$ we denote the number of expected moves for the solitaire Memory game using an optimal strategy. Furthermore, for $k < 0$ or $k > n$, we define $e_{n,k} = 0$.*

For each move there are four possibilities when using the optimal strategy from Lemma 4.1:

1. A known card is turned over and the corresponding pair is collected. This decreases n by 1 and k by 1.
2. An unknown card is turned over, but then the next card matches this card – which means that this pair is collected. This decreases n by 1 and does not change k.
3. An unknown card is turned over, but then the next card matches one of the previously known k cards. This requires one extra move to collect the new pair. After this, n is decreased by 1 and k does not change.
4. An unknown card is turned over and the next card does not match one any known card. This does not change n and increases k by 2.

Therefore, we have the following Lemma:

Lemma 6.1. *The recurrence relation for the expected number of moves $e_{n,k}$ for solitaire Memory with an optimal strategy for $n \ge k \ge 0$ with $e_{0,0} = 0$ is given by:*

$$
\begin{aligned}
e_{n,k} = & \frac{k}{2n-k}\left(1 + e_{n-1,k-1}\right) \\
& + \frac{2(n-k)}{2n-k}\Big(\frac{1}{2n-k-1}\left(1 + e_{n-1,k}\right) \\
& + \frac{k}{2n-k-1}\left(2 + e_{n-1,k}\right) \\
& + \frac{2(n-k-1)}{2n-k-1}\left(1 + e_{n,k+2}\right)\Big)
\end{aligned}
\tag{6.1}
$$

In the following table, we list some values for $e_{n,k}$ for games with small n. If a position is not possible using our optimal strategy, then we list a 0 as an entry.

7	$\frac{44141}{4095}$	$\frac{42206}{4095}$	$\frac{29197}{2970}$	$\frac{23123}{2475}$	$\frac{398}{45}$	$\frac{449}{54}$	$\frac{31}{4}$	7
6	$\frac{31748}{3465}$	$\frac{30143}{3465}$	$\frac{12952}{1575}$	$\frac{488}{63}$	$\frac{152}{21}$	$\frac{47}{7}$	6	0
5	$\frac{793}{105}$	$\frac{248}{35}$	$\frac{557}{84}$	$\frac{43}{7}$	$\frac{17}{3}$	5	0	0
4	$\frac{622}{105}$	$\frac{577}{105}$	$\frac{226}{45}$	$\frac{23}{5}$	4	0	0	0
3	$\frac{13}{3}$	$\frac{58}{15}$	$\frac{7}{2}$	3	0	0	0	0
2	$\frac{8}{3}$	$\frac{7}{3}$	2	0	0	0	0	0
1	1	1	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0
n\k	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**

In the following subsections, we will give an upper bound of $1.75 \cdot n$ for $e_{n,0}$ (Subsection 6.1), a lower bound of $1.5 \cdot n - 1$ for $e_{n,0}$ (Subsection 6.2), and general upper and lower bounds of $1.5n - 0.5k - 1 \leq e_{n,k} \leq 2n - k$ (Subsection 6.3). We note that the lower bound of $1.5 \cdot n - 1$ for $e_{n,0}$ will be needed for the proof of the lower bound of $1.5n - 0.5k - 1$ for $e_{n,k}$.

6.1 An Upper Bound for $e_{n,0}$

THEOREM 6.1. *Using an optimal strategy, for the expected number of moves $e_{n,0}$ for the solitaire Memory game with $n > 0$ pairs the following upper bound holds:*

$$e_{n,0} < 1.75 \cdot n \; . \tag{6.2}$$

Proof. We again consider an amortized analysis for the solitaire Memory game: At the end of a game, each pair is collected. When using an optimal strategy, every pair was collected in one of the following ways:

- 1.0–pair: The pair was collected during one move, turning each card over once – requiring two turn-overs in total, or 1 move in average for the pair.
- 2.0–pair: Both cards were turned over during different moves, and another move was needed to collect the pair. Therefore, each card was turned over twice, requiring four turn-overs in total, or 2 moves in average for the pair.
- 1.5–pair: Both cards were turned over during different moves, but collected in the second move. Therefore, the first known card was turned over twice and the second card was turned over once, requiring three turn-overs in total, or 1.5 moves in average for the pair.

Each time a a new card is turned over at the beginning of a new move, there is a chance of $\frac{k}{2n-k}$ for a 1.5–pair and a chance of $\frac{2(n-k)}{2n-k} \cdot \frac{k}{2n-k-1}$ for a 2.0–pair to appear. However, for $n \geq k \geq 0$ it holds that the chance for a 1.5–pair is at least as high as the chance for a 2.0–pair, since $\left(\frac{k}{2n-k}\right) - \left(\frac{2(n-k)}{2n-k} \cdot \frac{k}{2n-k-1}\right) = \frac{(k-1)k}{(k-2n)(k-2n+1)} = \frac{(k^2-k)}{(2n-k)(2n-k-1)} \geq 0$.

This means that in total, the number of expected 1.5–pairs is at least as large as the number of expected 2.0–pairs. Since the number of expected 1.0–pairs is strictly larger than zero, the expected number of turn-overs per card is less than $(2 + 1.5)/2 = 1.75$. ☐

6.2 A Lower Bound for $e_{n,0}$

THEOREM 6.2. *Using an optimal strategy, for the expected number of moves $e_{n,0}$ for the solitaire Memory game with $n > 0$ pairs the following lower bound holds:*

$$e_{n,0} \geq 1.5 \cdot n - 1 \; . \tag{6.3}$$

A straight-forward (but not very surprising) lower bound for the number of moves for the solitaire Memory game is $1 \cdot n$, since every pair is either a 1.0–pair or 1.5–pair or a 2.0–pair. However, the total expected number of 1.0–pairs can be bounded as follows:

LEMMA 6.2. *For a solitaire Memory game with n pairs, the expected number of collected 1.0–pairs is at most 1.*

With Lemma 6.2 we can now prove Theorem 6.2:

Proof. (*Theorem 6.2*)
Using the optimal strategy from Lemma 4.1, every collected pair is either a 1.0–, a 1.5–, or a 2.0–pair. With Lemma 6.2, the expected number of 1.0–pairs per game is at most 1. This results in the claimed lower bound of $1.5 \cdot n - 1$. ☐

Proof. (*Lemma 6.2*)
We can assume w.l.o.g. that each of the $2n$ cards is placed at a position i from $i = 1$ to $i = 2n$ and that for a 1.0–pair to be collected, both cards of the 1.0–pair must have adjacent positions i and $i + 1$, with $1 \leq i \leq 2n - 1$. Different strategies could employ different turn-over strategies, but since the cards are shuffled and placed uniformly at random, the assumption holds w.l.o.g. Adjacent positions are a necessary condition, but not a sufficient one, since both cards could be turned over in different moves. Nonetheless, we can use it for an upper bound on the number of 1.0–pairs.

The chance for the card at position i to be a matching card with the same label to the one at position

$i+1$ is $\frac{1}{2n-1}$, with $1 \leq i \leq 2n-1$. We can use this for a rough upper bound on the number of expected collected 1.0–pairs, with $\sum_{i=1}^{2n-1} \frac{1}{2n-1} = 1$. □

6.3 General Upper and Lower Bounds for $e_{n,k}$

In the previous subsections we considered the solitaire Memory game with $k = 0$ known cards, i.e., we focussed our investigations on the beginning of a new game. For a more specific understanding of the game process, we now consider the following question: How many moves are needed in expectation if we have already played some moves, have n pairs left on the table and so far only know the position of k cards with different labels? The following Theorems 6.3 and 6.4 give upper and lower bounds on the expected number $e_{n,k}$ of moves with an optimal strategy for $0 \leq k \leq n$:

THEOREM 6.3. *Using an optimal strategy, for the expected number of moves $e_{n,k}$ for the solitaire Memory game with $n > 0$ pairs the following upper bound holds for $n \geq k \geq 0$:*

$$e_{n,k} \leq 2n - k \; . \tag{6.4}$$

THEOREM 6.4. *Using an optimal strategy, for the expected number of moves $e_{n,k}$ for the solitaire Memory game with $n > 0$ pairs the following lower bound holds for $n \geq k \geq 0$:*

$$e_{n,k} \geq 1.5n - 0.5k - 1 \; . \tag{6.5}$$

We will prove Theorem 6.3 and 6.4 by induction. The idea of the proofs is visualized in Figure 2: To prove the bounds for a specific $e_{n,k}$ by induction, we need the values of $e_{n-1,k-1}$, $e_{n-1,k}$ and $e_{n,k+2}$ (cf. (6.1)). Basically, we work our way "up" line by line starting at $n = 0$, then for $n = 1$, and so on. Considering a new line, we work our way from "right" to "left". While we can assume that the induction hypothesis holds for all smaller n (i.e., the lines below), we need starting values for $e_{n,n-1}$ and $e_{n,n}$ (i.e., when "starting" a new line):

LEMMA 6.3. *Using an optimal strategy, the expected number of moves $e_{n,n}$ for the solitaire Memory game is exactly:*

$$e_{n,n} = n \; . \tag{6.6}$$

LEMMA 6.4. *Using an optimal strategy, the expected number of moves $e_{n,n-1}$ for the solitaire Memory game is exactly:*

$$e_{n,n-1} = n + \frac{n-1}{n+1} \; . \tag{6.7}$$

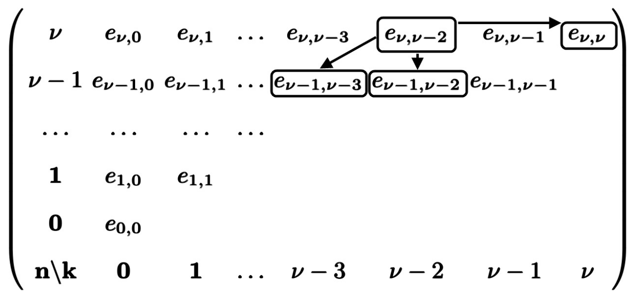

Figure 2: To calculate the value of $e_{\nu,\nu-2}$, we need the values of $e_{\nu-1,\nu-3}$, $e_{\nu-1,\nu-2}$, and $e_{\nu,\nu}$. The induction hypothesis gives us values for the lines below, but when we start a new line from the right, we need values for the two right-most entries, i.e., $e_{\nu,\nu-1}$ from Lemma 6.4 and $e_{\nu,\nu}$ from Lemma 6.3, to finish the calculation of the whole line of values.

We start with the proof of Lemma 6.3, which will be needed for the proof of Lemma 6.4:

Proof. (Lemma 6.3)
If the player has n pairs of cards left on the table and knows the position of $k = n$ cards with different labels, then she will finish the game in exactly n moves using the optimal strategy from Lemma 4.1: She will open an unknown card, know the position of the matching card with the same label, and collect the pair – this will be repeated n times.

This can also be proven by induction: The base case holds for $n = 0$ and $n = 1$, since for these game situations, the player needs $e_{0,0} = 0$ and $e_{1,1} = 1$ moves respectively. Let us now assume that $e_{\mu,\mu} = \mu$ holds for all μ with $0 \leq \mu < n$. Using the definition of $e_{n,k}$ from (6.1) completes the proof:

$$\begin{aligned} e_{n,n} =& \frac{n}{2n-n}\left(1 + e_{n-1,n-1}\right) \\ &+ \frac{2(n-n)}{2n-n}\Big(\frac{1}{2n-n-1}\left(1 + e_{n-1,n}\right) \\ &+ \frac{n}{2n-n-1}\left(2 + e_{n-1,n}\right) \\ &+ \frac{2(n-n-1)}{2n-n-1}\left(1 + e_{n,n+2}\right)\Big) \\ =& \frac{n}{2n-n}\left(1 + e_{n-1,n-1}\right) \\ =& 1 + e_{n-1,n-1} = n \; . \end{aligned} \tag{6.8}$$

□

Proof. (Lemma 6.4)
We prove Lemma 6.4 in two steps. First we use the definition of $e_{n,k}$ from (6.1) and $e_{n,n} = n$ from Lemma

6.3 to prove the following Equation 6.9, which will then be used to complete the proof of Lemma 6.4 by induction:
(6.9)

$$
\begin{aligned}
e_{n,n-1} =& \frac{n-1}{n+1}\left(1+e_{n-1,n-2}\right) \\
&+ \frac{2}{n+1}\left(\frac{1}{n}\left(1+e_{n-1,n-1}\right)\right. \\
&+ \left.\frac{n-1}{n}\left(2+e_{n-1,n-1}\right)+0\right) \\
=& \frac{n-1}{n+1}e_{n-1,n-2}+\frac{n-1}{n+1}+\frac{2}{n+1} \\
&\left(\frac{1}{n}\left(1+(n-1)\right)+\frac{n-1}{n}\left(2+(n-1)\right)\right) \\
=& \frac{n-1}{n+1}e_{n-1,n-2}+\frac{1}{n+1} \\
&\left((n-1)+2+2\frac{n-1}{n}(n+1)\right) \\
=& n+\frac{n-1}{n+1}e_{n-1,n-2}+1+2\frac{n-1}{n} \\
=& \frac{n-1}{n+1}e_{n-1,n-2}+3-\frac{2}{n}
\end{aligned}
$$

We can now prove $e_{n,n-1} = n + \frac{n-1}{n+1}$ by induction using $e_{1,0} = 1$ as the base case and Equation (6.9). Let us now assume that the induction hypothesis of $e_{\mu,\mu-1} = \mu + \frac{\mu-1}{\mu+1}$ holds for all μ with $1 \le \mu \le n$:
(6.10)

$$
\begin{aligned}
e_{n+1,n} =& \frac{n}{n+2}e_{n,n-1}+3-\frac{2}{n+1} \\
=& \frac{n}{n+2}\left(n+\frac{n-1}{n+1}\right)+3-\frac{2}{n+1} \\
=& \frac{1}{n+1}\left(n^2+\frac{n^2-n}{n+1}+3n+6-2\frac{n+2}{n+1}\right) \\
=& (n+1)+\frac{1}{n+2}\left(n^2+3n+6\right. \\
&+\left.\frac{n^2-n-n-4}{n+1}-(n+1)(n+2)\right) \\
=& (n+1)+\frac{1}{n+2}\left(6-2+\frac{(n+1)(n-4)}{n+1}\right) \\
=& (n+1)+\frac{n}{n+2}\ .
\end{aligned}
$$

□

We can now prove the Theorems 6.3 and 6.4:

Proof. (Theorem 6.3)
We use induction for the proof: Theorem 6.3 holds for $k = n$ with $e_{n,n} = n \le 2n - n = n$ (Lemma 6.3) and for $k = n-1$ with $e_{n,n-1} = n + \frac{n-1}{n+1} \le 2n-(n+1) = n+1$ (Lemma 6.4).

For fixed $0 \le k \le n$ let us now assume that $e_{\nu,\mu} \le 2\nu-\mu$ holds

- for all $\nu < n$ and $0 \le \mu \le \nu$
- as well as for $\nu = n$ for all $\mu \ge k+1$

Then we can bound $e_{n,k}$ from above:
(6.11)

$$
\begin{aligned}
e_{n,k} =& \frac{k}{2n-k}\left(1+e_{n-1,k-1}\right) \\
&+ \frac{2(n-k)}{2n-k}\left(\frac{1}{2n-k-1}\left(1+e_{n-1,k}\right)\right. \\
&+ \frac{k}{2n-k-1}\left(2+e_{n-1,k}\right) \\
&+\left.\frac{2(n-k-1)}{2n-k-1}\left(1+e_{n,k+2}\right)\right) \\
\le& \frac{k}{2n-k}\left(2n-k\right) \\
&+ \frac{2(n-k)}{2n-k}\frac{1}{2n-k-1}\left((1+2n-2-k)+\right. \\
&\left.k(2+2n-2-k)+2(n-k-1)(1+2n-k-2)\right) \\
\le& k+2(n-k)\frac{1}{2n-k-1}\left(1+k+2(n-k-1)\right) \\
\le& k+\frac{2(n-k)}{2n-k-1}\left(2n-k-1\right) \\
\le& k+2n-2k = 2n-k.
\end{aligned}
$$

□

Proof. (Theorem 6.4)
Again, we can prove Theorem 6.4 by induction. The base case holds with:

- $k = n$: $e_{n,n} = n > 1.5n - 0.5n - 1$ (Lemma 6.3)
- $k = n-1$: $e_{n,n-1} = n + \frac{n-1}{n+1} > 1.5n - 0.5n - 0.5 = n - 0.5$ (Lemma 6.4)
- $k = 0$: $e_{n,0} > 1.5n - 1$ (Theorem 6.2)

For fixed $0 \le k \le n$ we assume that $e_{\nu,\mu} \ge 1.5\nu-0.5\mu-1$ holds

- for all $\nu < n$ and $0 \le \mu \le \nu$
- as well as for $\nu = n$ for all $\mu \ge k+1$

Then we can bound $e_{n,k}$ from below:

$$
\begin{aligned}
e_{n,k} \geq & \frac{k}{2n-k}\left(1+\frac{3}{2}(n-1)-\frac{1}{2}(k-1)-1\right) \\
& + \frac{2(n-k)}{2n-k}\frac{1}{2n-k-1}\Bigg((2k+1) \\
& + (k+1)\left(\frac{3}{2}(n-1)-\frac{1}{2}k-1\right) \\
& + 2(n-k-1)\left(1+\frac{3}{2}n-\frac{1}{2}(k-2)-1\right)\Bigg) \\
= & \frac{k}{2n-k}\left(\frac{3}{2}n-\frac{1}{2}k-1\right) \\
& + \frac{2(n-k)}{2n-k}\frac{1}{2n-k-1}\Bigg((2n-k-1) \\
& \left(\frac{3}{2}n-\frac{1}{2}k-1\right)+(2k+1) \\
& + (k+1)-\left(-\frac{3}{2}\right)\Bigg) \\
= & \left(\frac{3}{2}n-\frac{1}{2}k-1\right) \\
& + \frac{2(n-k)}{2n-k}\frac{1}{2n-k-1}\left(\frac{1}{2}k-\frac{1}{2}\right)
\end{aligned} \tag{6.12}
$$

$$
\begin{aligned}
= & \left(\frac{3}{2}n-\frac{1}{2}k-1\right) \\
& + \frac{n-k}{2n-k}\frac{1}{2n-k-1}(k-1) \qquad .
\end{aligned} \tag{6.13}
$$

With (6.13) we can now see that

$$
e_{n,k} \geq \frac{3}{2}n-\frac{1}{2}k-1 \tag{6.14}
$$

holds for $1 \leq k \leq n$. The remaining case $k = 0$ holds due to Theorem 6.2, which completes the proof by induction. □

7 Numerical Data and Conjectures

The exact rational values for the expected number of moves $e_{n,0}$ for the solitaire Memory game can be calculated using the recurrence relation (6.1). The values from $e_{1,0}$ to $e_{5000,0}$ are plotted in Figure 3, while the average number of moves per pair $e_{n,0}/n$ can be seen in Figure 4. For solitaire Memory games in the tournament version (with 32 or 31) pairs, the expected number of needed moves with an optimal strategy are

$$
\begin{aligned}
e_{32,0} &= \frac{73212976706398781542803 86}{1432009307295085110 84225} \\
&\approx 51.1260 \approx 1.5977 \cdot 32
\end{aligned} \tag{7.1}
$$

$$
\begin{aligned}
e_{31,0} &= \frac{914636795802985883205078251}{18472920064106597929865025} \\
&\approx 49.5123 \approx 1.5972 \cdot 31
\end{aligned} \tag{7.2}
$$

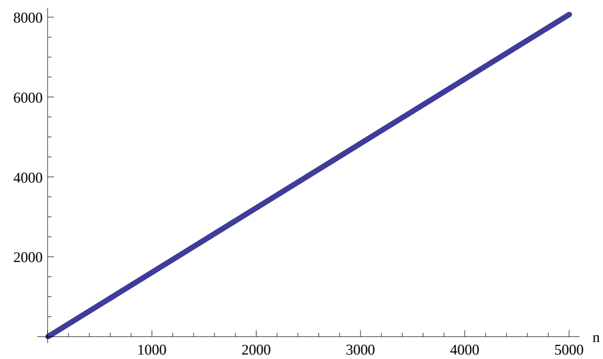

Figure 3: Expected number of moves $e_{n,0}$ for the solitaire Memory game with an optimal strategy. The value for $e_{5000,0}$ is ≈ 8068.01689.

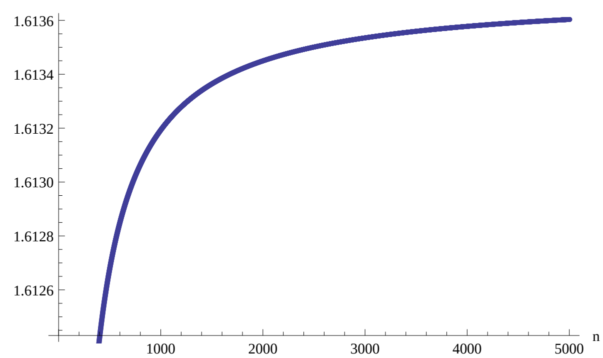

Figure 4: Expected number of moves for the solitaire Memory game with an optimal strategy – divided by n, i.e., $e_{n,0}/n$. The value for $e_{5000,0}/5000$ is ≈ 1.613603.

Observation 1. *For $1 \leq n \leq 5000$, the expected average number of needed moves per pair, i.e., $e_{n,0}/n$, is strictly monotonically increasing.*

Conjecture 1. *The expected average number of needed moves per pair to finish the solitaire Memory game, i.e., $e_{n,0}/n$, is strictly monotonically increasing.*

7.1 The Competitive Ratio of the Solitaire Memory Game If one considers the solitaire Memory game as an online problem (cf. [1]), then the optimal strategy from Lemma 4.1 competes against a fictional optimal offline player that already knows the labels of

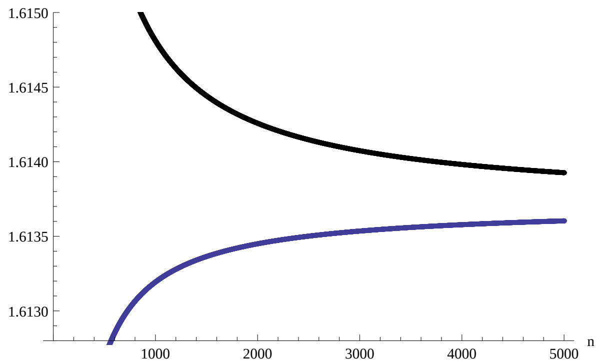

Figure 5: The upper black graph represents $e_{n,0}/(n-1)$, the lower blue one $e_{n,0}/n$. The value of $e_{5000,0}/(5000-1)$ ist ≈ 1.613926. We note that the function $e_{n,0}/(n-1)$ is strictly monotonically decreasing for $2 \leq n \leq 5000$.

the cards beforehand. With all information available, a pair can be collected in each move by the offline player, requiring $1 \cdot n$ moves in total.

Definition 7.1. *A strategy S has a competitive ratio of $c = \frac{c \cdot n}{1 \cdot n}$ for the solitaire Memory game with n pairs, if S finishes the solitaire Memory with at most $c \cdot n$ expected moves for all $n \in \mathbb{N}$.*

What competitive ratio can the optimal strategy from Lemma 4.1 achieve? With Theorem 6.1 it is at most $\frac{1.75 \cdot n}{1 \cdot n} = 1.75$. With $e_{5000,0}/5000 \approx 1.61360$ and Conjecture 1, one could believe 1.75 might an asymptotic upper bound. However, the data from Figure 6 leads to another Conjecture:

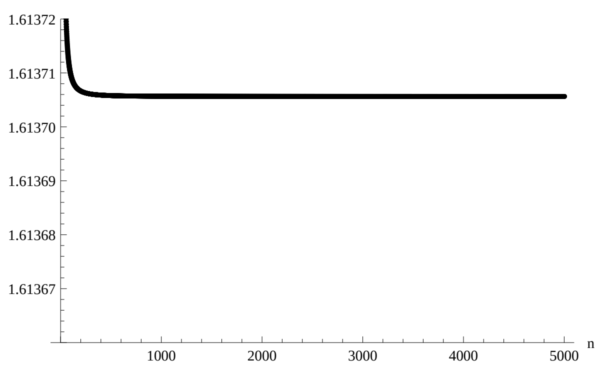

Figure 6: Expected number of additional moves for the solitaire Memory game when the number of pairs is increased by one, i.e., $e_{n,0} - e_{n-1,0}$. The value for $e_{5000,0} - e_{4999,0}$ is ≈ 1.613706

Observation 2. *For $12 \leq n \leq 5000$, the expected number of additional moves for the solitaire Memory game when the number of pairs is increased by one, i.e., $e_{n,0} - e_{n-1,0}$, is strictly monotonically decreasing. The first three values are $e_{3,0} - e_{2,0} = e_{2,0} - e_{1,0} = 5/3$ and $e_{1,0} - e_{0,0} = 1$. For $4 \leq n \leq 12$ the function $e_{n,0} - e_{n-1,0}$ has the following values (rounded to six places), which decrease and increase alternately:* 1.59048, 1.62857, 1.61010, 1.61676, 1.61374, 1.61458, 1.61408, 1.61413, 1.61401.

Conjecture 2. *For $n \geq 12$ the expected number of additional moves for the solitaire Memory game when the number of pairs is increased by one, i.e., $e_{n,0} - e_{n-1,0}$, is strictly monotonically decreasing.*

If Conjecture 1 and 2 hold, then $e_{n,0}/n$ must converge towards a constant C for $n \to \infty$. In particular, the validity of both conjectures would imply sharper upper and lower bounds for the competitive ratio C by calculating just one single pair of values $e_{n-1,0}$ and $e_{n,0}$ (cf. Figure 7). Using this idea and with the help of our calculations for $e_{4999,0}$ and $e_{5000,0}$ the following conjecture would be proven:

Conjecture 3. *The competitive ratio of an optimal strategy for the solitaire Memory game is between* 1.613603 *and* 1.613706.

For a comparison with our results obtained in Section 6, note the validity of the following corollary:

Corollary 7.1. *For $n > 200$ it holds that*

$$e_{n,0} = 1.625 \cdot (1 + \epsilon) \tag{7.3}$$

with $|\epsilon| < 0.08$.

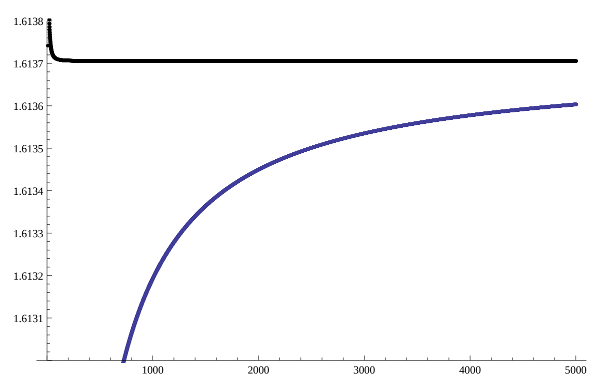

Figure 7: The graphs from Figure 4 and Figure 6 in one image. The upper black graph represents $e_{n,0} - e_{n-1,0}$, the lower blue one $e_{n,0}/n$.

Using an analog reasoning, we can come to similar conjectures for the expected number of 1.5– and 2.0–pairs and for moves where two cards with different and yet not known labels are turned over. Due to space constraints, we give the details in the appendix:

Conjecture 4. *The expected number of 1.5–pairs for the solitaire Memory game using the optimal strategy from Lemma 4.1 for* $n \to \infty$ *is between* $0.77251598 \cdot n$ *and* $0.77258872 \cdot n$.

Conjecture 5. *The expected number of 2.0–pairs multiplied by two for the solitaire Memory game using the optimal strategy from Lemma 4.1 for* $n \to \infty$ *is between* $0.45469078 \cdot n$ *and* $0.45482256 \cdot n$.

Conjecture 6. *The expected number moves where two cards with different and yet not known labels are turned over for the solitaire Memory game using the optimal strategy from Lemma 4.1 for* $n \to \infty$ *is between* $0.38625799 \cdot n$ *and* $0.38629436 \cdot n$.

However, if we consider games in progress where quite a few cards are known, e.g., $e_{n,n-k}$ with $k \in \mathcal{O}(\sqrt{n})$, the further expected number of needed moves per pair approaches one! The combination of Theorem 6.3 and Theorem 6.4 yields the following corollary:

Corollary 7.2. *Let* $k = k(n) \in o(n)$. *Then*

$$\lim_{n\to\infty} \frac{e_{n,n-k}}{n} = 1 \ . \tag{7.4}$$

Judging from numerical data (cf. Figure 7.1), we also conjecture that the following holds for **fixed** k:

$$\lim_{n\to\infty} (-n + e_{n,n-k}) = k \ . \tag{7.5}$$

We note that for $k = 0$ and $k = 1$, the Conjecture (7.5) is true (see Lemma 6.3 and 6.4).

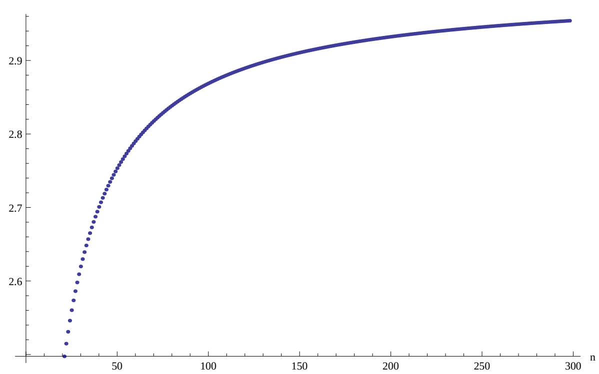

Figure 8: The graph of $-n + e_{n,n-3}$. We note that already for n up to 300, the value is quite close to 3. Note also that due to Theorem 6.3, the value can never go above $3 = n - (n-3) = -n + 2n - k$.

8 Variations of the Model

What happens if the next move is free each time after we collect a pair? In total, the player then gets n moves for free (one for each collected pair) – however the last one does not help her, because then all cards are collected already. Therefore this lowers the number of moves in the expected case by $n-1$ (and also in the deterministic case). The modified recurrence relation $e^f_{n,k}$ can be defined by

$$\begin{aligned} e^f_{n,k} =& \frac{k}{2n-k}\left(0 + e^f_{n-1,k-1}\right) \\ &+ \frac{2(n-k)}{2n-k}\Bigg(\frac{1}{2n-k-1}\left(0 + e^f_{n-1,k}\right) \\ &+ \frac{k}{2n-k-1}\left(1 + e^f_{n-1,k}\right) \\ &+ \frac{2(n-k-1)}{2n-k-1}\left(1 + e^f_{n,k+2}\right)\Bigg) \end{aligned} \tag{8.1}$$

with $e^f_{0,0} = 0$ and $e^f_{1,0} = 1 = e^f_{1,1}$ (all costs of 1–pairs, 1.5–pairs and 2–pairs get reduced by 1 – except for the last collected pair). For the tournament versions of Memory the expected number of moves in this model are thus (cf. (7.1) and (7.2)):

$$e^f_{32,0} \approx 20.1260 \ , \tag{8.2}$$

$$e^f_{31,0} \approx 19.5123 \ . \tag{8.3}$$

The recurrence relation 8.1 also describes the expected number of moves plus one with the strategy from Lemma 4.1 where no pair is collected. This gives another way to describe the value of $e_{n,0}$:

$$e_{n,0} = e^f_{n,0} + (n-1) \ . \tag{8.4}$$

8.1 Making the Game Easier

Many moves in the Memory game are required because a pair can only be collected if both cards of the pair are turned over in the same move, but not if they are turned over one after another in succinct moves. If this requirement would be dropped, then a pair could be collected if both cards are turned over in succinct turn overs. How many moves does the game now require? This change of the rules would eliminate all 2.0-pairs, leaving only 1.0–pairs (2 turn overs) and 1.5–pairs (3 turn overs). Since the number of 1.0–pairs is bounded from above by 1 even under this rule-change (cf. the proof of Lemma 6.2), the expected number $e^{easy}_{n,0}$ of moves for a new solitaire Memory game with n pairs under this change of rules can be bounded by:

$$1.5 \cdot n - 1 \le e^{easy}_{n,0} \le 1.5 \cdot n \ . \tag{8.5}$$

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9 Appendix: Additional Numerical Data

In this section we discuss further numerical data and conjectures for the optimal strategy from Lemma 4.1. One subsection each is dedicated to the expected number of 1.0–, 1.5–, and 2.0–pairs and other moves (i.e., where two cards with yet unknown labels are turned over), followed by a number of tables for small n and graphs for large n.

9.1 Expected Number of 1.0–Pairs

The expected number of 1.0–pairs for the solitaire Memory game using the optimal strategy from Lemma 4.1 is at most 1, see Lemma 6.2. The exact values $e^{1.0}_{n,k}$ can be calculated using the following recurrence relation with $e^{1.0}_{0,0} = 0$:

$$\begin{aligned} e^{1.0}_{n,k} = &\frac{k}{2n-k}\left(0 + e^{1.0}_{n-1,k-1}\right) \\ &+ \frac{2(n-k)}{2n-k}\Big(\frac{1}{2n-k-1}\left(1 + e^{1.0}_{n-1,k}\right) \\ &+ \frac{k}{2n-k-1}\left(0 + e^{1.0}_{n-1,k}\right) \\ &+ \frac{2(n-k-1)}{2n-k-1}\left(0 + e^{1.0}_{n,k+2}\right)\Big) \end{aligned} \tag{9.1}$$

The values for small n are listed in Table 3, while a graph for n up to 5000 can be seen in Figure 9.

Observation 3. *The expected number of 1.0–pairs for the solitaire Memory game using the optimal strategy from Lemma 4.1 is strictly monotonically decreasing for* $5 \le n \le 5000$.

9.2 Expected Number of 1.5–Pairs

The expected number of 1.5–pairs for the solitaire Memory game using the optimal strategy from Lemma 4.1 is at least as large as the expected number of 2.0–pairs, see the proof of Theorem 6.1. The exact values $e^{1.5}_{n,k}$ can be calculated using the following recurrence relation with $e^{1.5}_{0,0} = 0$:

$$\begin{aligned} e^{1.5}_{n,k} = &\frac{k}{2n-k}\left(1 + e^{1.5}_{n-1,k-1}\right) \\ &+ \frac{2(n-k)}{2n-k}\Big(\frac{1}{2n-k-1}\left(0 + e^{1.5}_{n-1,k}\right) \\ &+ \frac{k}{2n-k-1}\left(0 + e^{1.5}_{n-1,k}\right) \\ &+ \frac{2(n-k-1)}{2n-k-1}\left(0 + e^{1.5}_{n,k+2}\right)\Big) \end{aligned} \tag{9.2}$$

The values for small n are listed in Table 4, while a graph for n up to 5000 can be seen in Figure 11.

Observation 4. *The expected number of 1.5–pairs for the solitaire Memory game using the optimal strategy from Lemma 4.1 divided by n, i.e.,* $e^{1.5}_{n,0}/n$*, is strictly monotonically increasing for* $3 \le n \le 5000$.

Observation 5. *For* $12 \le n \le 5000$*, the expected number of additional 1.5–pairs for the solitaire Memory game when the number of pairs is increased by one, i.e.,* $e^{1.5}_{n,0} - e^{1.5}_{n-1,0}$*, is strictly monotonically decreasing. The first three values are* $e^{1.5}_{3,0} - e^{1.5}_{2,0} = 8/15$*,* $e^{1.5}_{2,0} - e^{1.5}_{1,0} = 4/3$*, and* $e^{1.5}_{1,0} - e^{1.5}_{0,0} = 0$*. For* $4 \le n \le 12$ *the function* $e^{1.5}_{n,0} - e^{1.5}_{n-1,0}$ *has the following values (rounded to six places), which decrease and increase alternately:* $0.87619, 0.73651, 0.78846, 0.76848, 0.77531, 0.77257, 0.77335, 0.77293, 0.77298$.

Using an analog reasoning as for Conjecture 3 leads us to believe that:

Conjecture 4. *The expected number of 1.5–pairs for the solitaire Memory game using the optimal strategy from Lemma 4.1 for* $n \to \infty$ *is between* $0.77251598 \cdot n$ *and* $0.77258872 \cdot n$.

9.3 Expected Number of 2.0–Pairs

The 2.0–pairs are essentially the type of pairs that make the solitaire Memory game "expensive" in terms of the number of moves needed. In the 2-player variant, turning over a card with an already known label at the end of the move (which is different from the first turn over) is the worst that can happen, because then the opponent can immediately collect a pair. Their expected amount multiplied by two using the optimal strategy from Lemma 4.1 can be calculated using the following recurrence relation with $e^{2.0}_{0,0} = 0$.

$$\begin{aligned} e^{2.0}_{n,k} = &\Bigg(\frac{k}{2n-k}\left(0 + e^{2.0}_{n-1,k-1}\right) \\ &+ \frac{2(n-k)}{2n-k}\Big(\frac{1}{2n-k-1}\left(0 + e^{2.0}_{n-1,k}\right) \\ &+ \frac{k}{2n-k-1}\left(2 + e^{2.0}_{n-1,k}\right) \\ &+ \frac{2(n-k-1)}{2n-k-1}\left(0 + e^{2.0}_{n,k+2}\right)\Big)\Bigg) \end{aligned} \tag{9.3}$$

The values for small n are listed in Table 5, while a graph for n up to 5000 can be seen in Figure 15.

Observation 6. *The expected number of 2.0–pairs multiplied by two for the solitaire Memory game using the optimal strategy from Lemma 4.1 divided by n, i.e.,* $e^{2.0}_{n,0}/n$*, is strictly monotonically increasing for* $2 \le n \le 5000$.

Observation 7. *For $2 \le n \le 5000$, the expected number of additional 2.0–pairs multiplied by two for the solitaire Memory game when the number of pairs is increased by one, i.e., $e^{2.0}_{n,0} - e^{2.0}_{n-1,0}$, is strictly monotonically decreasing. The first two values are $e^{2.0}_{1,0} - e^{2.0}_{0,0} = 0$ and $e^{2.0}_{2,0} - e^{2.0}_{1,0} = 0$.*

Using an analog reasoning as for Conjecture 3 leads us to believe that:

Conjecture 5. *The expected number of 2.0–pairs multiplied by two for the solitaire Memory game using the optimal strategy from Lemma 4.1 for $n \to \infty$ is between* $0.45469078 \cdot n$ *and* $0.45482256 \cdot n$.

9.4 Expected Number of other Moves

So far, we only looked at the moves generating 1.0–, 1.5–, and 2.0–pairs. One thing these three types have in common, is that a new matching pair of cards was found and collected. We now look at moves where two cards with different and not yet seen labels have been turned over. For the sake of simplicity, we call them "other" moves and denote their expected amount with $e^{other}_{n,k}$. Their expected amount using the optimal strategy from Lemma 4.1 can be calculated using the following recurrence relation with $e^{other}_{0,0} = 0$:

$$e^{other}_{n,k} = \Bigg(\frac{k}{2n-k}\left(0 + e^{other}_{n-1,k-1}\right) + \frac{2(n-k)}{2n-k}\Big(\frac{1}{2n-k-1}\left(0 + e^{other}_{n-1,k}\right) + \frac{k}{2n-k-1}\left(0 + e^{other}_{n-1,k}\right) + \frac{2(n-k-1)}{2n-k-1}\left(1 + e^{other}_{n,k+2}\right) \Big) \Bigg) \tag{9.4}$$

The values for small n are listed in Table 6, while a graph for n up to 5000 can be seen in Figure 19. This gives us one more way to calculate $e_{n,0}$:

$$e_{n,0} = e^{1.0}_{n,0} + e^{1.5}_{n,0} + e^{2.0}_{n,0} + e^{other}_{n,0} \tag{9.5}$$

Another possibility would be to use the fact that

$$n = e^{1.0}_{n,0} + e^{1.5}_{n,0} + \frac{1}{2} e^{2.0}_{n,0} \tag{9.6}$$

since every pair is either a 1.0–, a 1.5–, or a 2.0–pair, which leads to the following equation:

$$\frac{1}{2} e^{2.0}_{n,0} + e^{other}_{n,0} = e_{n,0} - n \tag{9.7}$$

Observation 8. *The expected number of other moves for the solitaire Memory game using the optimal strategy from Lemma 4.1 divided by n, i.e., $e^{other}_{n,0}/n$, is strictly monotonically increasing for $2 \le n \le 5000$.*

Observation 9. *For $12 \le n \le 5000$, the expected number of additional other moves for the solitaire Memory game when the number of pairs is increased by one, i.e., $e^{other}_{n,0} - e^{other}_{n-1,0}$, is strictly monotonically decreasing. The first two values are $e^{other}_{1,0} - e^{other}_{0,0} = 0$ and $e^{other}_{2,0} - e^{other}_{1,0} = \frac{2}{3}$. For $3 \le n \le 12$ the function $e^{other}_{n,0} - e^{other}_{n-1,0}$ has the following values (rounded to six places), which decrease and increase alternately:* 0.26667, 0.43810, 0.36825, 0.39423, 0.38424, 0.38766, 0.38629, 0.38668, 0.38647, 0.38649.

Using an analog reasoning as for Conjecture 3 leads us to believe that:

Conjecture 6. *The expected number of other moves for the solitaire Memory game using the optimal strategy from Lemma 4.1 for $n \to \infty$ is between* $0.38625799 \cdot n$ *and* $0.38629436 \cdot n$.

9.5 List of Tables and Graphs

n	exact value	approx. value
1	1	1
2	4/3	1.33333333
3	13/9	1.44444444
4	311/210	1.48095238
5	793/525	1.51047619
6	15874/10395	1.52708033
7	44141/28665	1.53989185
8	279121/180180	1.54912310
9	2145301/1378377	1.55639640
10	113643872/72747675	1.56216500
11	27863617/17782765	1.56688889
12	3153940403/2007835830	1.57081588
13	4891415443/3107364975	1.57413612
14	803050778/509233725	1.57697878
15	3448742150057/2183521465125	1.57944046
16	57096904850717/36100888223400	1.58159280
17	40492173334027/25571462491575	1.58349071
18	64379576011276/40613499251325	1.58517678
19	5033543598555223/3172365552631275	1.58668461
20	883832754109573/556555360110750	1.58804104

Table 1: Expected number of moves divided by n (i.e., $e_{n,0}/n = (e^{1.0}_{n,0} + e^{1.5}_{n,0} + e^{2.0}_{n,0} + e^{other}_{n,0})/n = (e^{f}_{n,0} + (n-1))/n$) for the solitaire Memory game with $1 \le n \le 20$ using the optimal strategy from Lemma 4.1.

n	exact value	approx. value
1	1	1
2	8/3	2.666666667
3	13/3	4.333333333
4	622/105	5.923809524
5	793/105	7.552380952
6	31748/3465	9.162481962
7	44141/4095	10.77924298
8	558242/45045	12.39298479
9	2145301/153153	14.00756760
10	227287744/14549535	15.62165004
11	27863617/1616615	17.23577784
12	6307880806/334639305	18.84979054
13	4891415443/239028075	20.46376955
14	1606101556/72747675	22.07770291
15	3448742150057/145568097675	23.69160692
16	114193809701434/4512611027925	25.30548478
17	40492173334027/1504203675975	26.91934209
18	128759152022552/4512611027925	28.53318206
19	5033543598555223/166966608033225	30.14700758
20	1767665508219146/55655536011075	31.76082084
21	20770032350719091/622330084487475	33.37462364
22	251201587269224572/7179564145428675	34.98841743
23	718286836964233567/19624141997505045	36.60220340
24	12295788072452732726/321744653680024575	38.21598256
25	42388113181280162029/1064232316018542825	39.82975572
26	4013603877497781469328/96845140757687397075	41.44352361
27	221004117084580646223401/5132792460157432044975	43.05728681
28	229287207222988691113978/5132792460157432044975	44.67104583
29	428054555268798632437/9248274702986364045	46.28480111
30	14505346581223189068650428/302834755149288490653525	47.89855304
31	914636795802985883205078251/18472920064106597929865025	49.51230193
32	7321297670639878154280386/143200930729508511084225	51.12604809
33	5209935600259851814752631/98785668792013892673075	52.73979175
34	1223137958164950353131429064/22503375350820764750926485	54.35353315
35	2099087566698655807690209517/37505625584701274584877475	55.96727248
36	459997313126900608532427858082/7988698249541371486578902175	57.58100992
37	1816889164606323279775720721981/30693419590343164132645255725	59.19474561
38	193781330401024147822315278916/3186748482057486986449507425	60.80847971
39	11440965466380244794544587499309/183283562696620608677795955615	62.42221233
40	32452074708112279917307914308093518/506779050856155982994105817275475	64.03594358
41	33269879261526852636262564323130991/506779050856155982994105817275475	65.64967357
42	8487833122027697923633162362535984736/126187983663182839765532348501593275	67.26340239
43	8691466170467351829769341074317848493/126187983663182839765532348501593275	68.87713012
44	39533773737363401213973443340997970/560835482947479287846810437784859	70.49085684
45	525494573903160586725743237859075983/7287949737847678610728344592239975	72.10458261
46	14032380470397378107619119294050660444/190351365186835131171735237570200025	73.71830749
47	282011249278755686734569374441364072763/3743576848674424246377459672213933825	75.33203155
48	864157039218989082062936221011816689302/11230730546023272739132379016641801475	76.94575484
49	17116238255318841282388878334413073037227/217876172592851491139168152922850948615	78.55947739
50	29113049668368192296518806200159569812632/363126954321419151898613588204751581025	80.17319927

Table 2: Expected number of moves (i.e., $e_{n,0} = e_{n,0}^{1.0} + e_{n,0}^{1.5} + e_{n,0}^{2.0} + e_{n,0}^{other} = e_{n,0}^{f} + (n-1)$) for the solitaire Memory game with $1 \leq n \leq 50$ using the optimal strategy from Lemma 4.1.

n	exact value	approx. value
1	1	1
2	2/3	0.6666666667
3	11/15	0.7333333333
4	74/105	0.7047619048
5	223/315	0.7079365079
6	2438/3465	0.7036075036
7	31649/45045	0.7026085026
8	31586/45045	0.7012099012
9	536297/765765	0.7003414886
10	10178618/14549535	0.6995837324
11	10169963/14549535	0.6989888680
12	233741674/334639305	0.6984884038
13	61474019/88062975	0.6980688422
14	3502216282/5019589575	0.6977096891
15	101519128079/145568097675	0.6973995656
16	165572169206/237505843575	0.6971288231
17	3144795740969/4512611027925	0.6968904968
18	628768329766/902522205585	0.6966790688
19	3751310092403/5386019613975	0.6964902398
20	16608897635558/23852372576175	0.6963205686

Table 3: Expected number of 1.0–pairs (i.e., $e^{1.0}_{n,0}$) for the solitaire Memory game with $1 \leq n \leq 20$ using the optimal strategy from Lemma 4.1.

n	exact value	approx. value
1	0	0
2	4/3	1.333333333
3	28/15	1.866666667
4	96/35	2.742857143
5	1096/315	3.479365079
6	14788/3465	4.267821068
7	15124/3003	5.036297036
8	261784/45045	5.811610612
9	5041936/765765	6.584181831
10	5097556/692835	7.357532457
11	118294508/14549535	8.130466575
12	31362544/3522519	8.903442111
13	1798932264/185910725	9.676323214
14	52450568908/5019589575	10.44917480
15	85977016012/7661478825	11.22198703
16	1640234665288/136745788725	11.99477279
17	329228105216/25786348731	12.76753482
18	1971032473156/145568097675	13.54027774
19	8753823398116/611599296825	14.31300435
20	2518811026445312/166966608033225	15.08571717

Table 4: Expected number of 1.5–pairs (i.e., $e^{1.5}_{n,0}$) for the solitaire Memory game with $1 \leq n \leq 20$ using the optimal strategy from Lemma 4.1.

n	exact value	approx. value
1	0	0
2	0	0
3	4/5	0.8000000000
4	116/105	1.104761905
5	512/315	1.625396825
6	72/35	2.057142857
7	113612/45045	2.522188922
8	116/39	2.974358974
9	875768/255255	3.430953360
10	56536112/145495355	3.885767621
11	63160828/14549535	4.341089114
12	534992204/111546435	4.796138971
13	46985648/8947575	5.251215888
14	5728587544/1003917915	5.706231016
15	14236159772/2310604725	6.161226807
16	29856322554764/4512611027925	6.616196781
17	31909346641912/4512611027925	7.071149373
18	11320766790656/1504203675975	7.526086375
19	1332562304162228/166966608033225	7.981010813
20	1408517701540564/166966608033225	8.435924513

Table 5: Expected number of 2.0–pairs multiplied by two (i.e., $e^{2.0}_{n,0}$) for the solitaire Memory game with $1 \leq n \leq 20$ using the optimal strategy from Lemma 4.1.

n	exact value	approx. value
1	0	0
2	2/3	0.6666666667
3	14/15	0.9333333333
4	48/35	1.371428571
5	548/315	1.739682540,
6	7394/3465	2.133910534
7	7562/3003	2.518148518
8	130892/45045	2.905805306
9	2520968/765765	3.292090916
10	2548778/692835	3.678766229
11	59147254/14549535	4.065233288
12	15681272/3522519	4.451721055
13	899466132/185910725	4.838161607
14	26225284454/5019589575	5.224587401
15	42988508006/7661478825	5.610993515
16	820117332644/136745788725	5.997386393
17	164614052608/25786348731	6.383767408
18	985516236578/145568097675	6.770138872
19	4376911699058/611599296825	7.156502177
20	1259405513222656/166966608033225	7.542858587

Table 6: Expected number of moves (i.e., $e^{other}_{n,0}$) where no card with a previously known label is turned over for the solitaire Memory game with $1 \leq n \leq 10$ using the optimal strategy from Lemma 4.1.

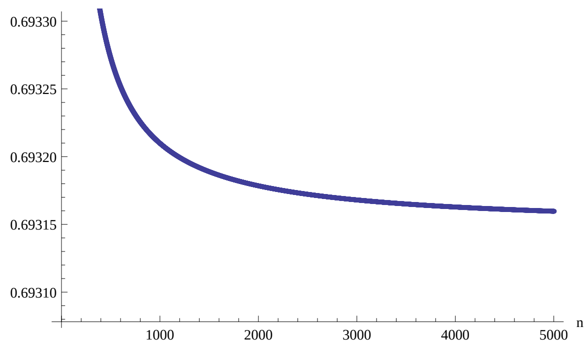

Figure 9: Expected number of 1.0–pairs for the solitaire Memory game using the optimal strategy from Lemma 4.1. The value for $n = 5000$ is ≈ 0.69316.

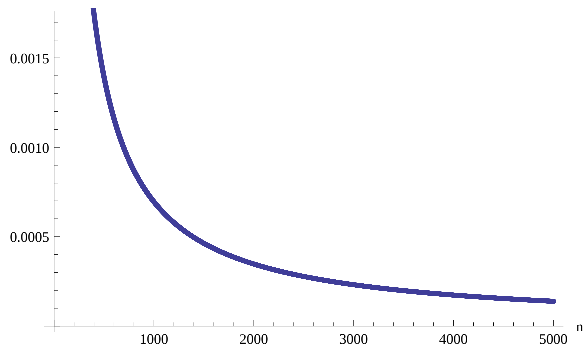

Figure 10: Expected number of 1.0–pairs for the solitaire Memory game using the optimal strategy from Lemma 4.1 – divided by n. The value for $n = 5000$ is ≈ 0.00013863.

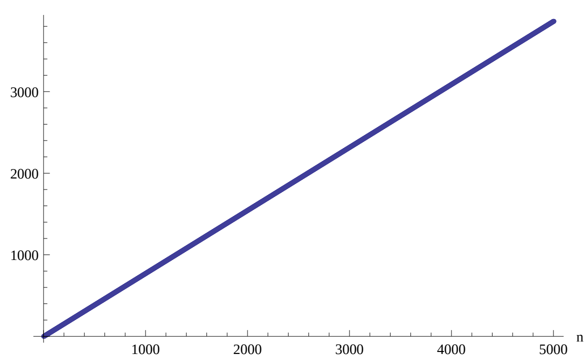

Figure 11: Expected number of 1.5–pairs for the solitaire Memory game using the optimal strategy from Lemma 4.1. The value for $n = 5000$ is ≈ 3862.5799.

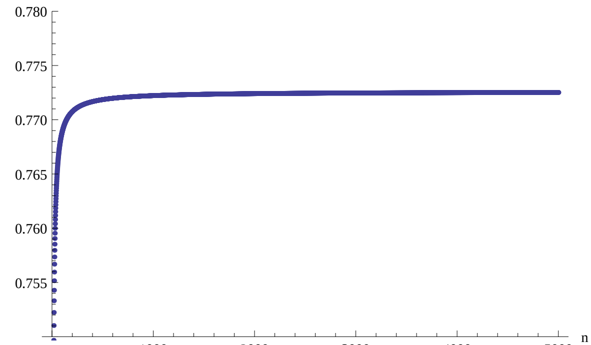

Figure 12: Expected number of 1.5–pairs for the solitaire Memory game using the optimal strategy from Lemma 4.1 – divided by n. The value for $n = 5000$ is ≈ 0.77251598.

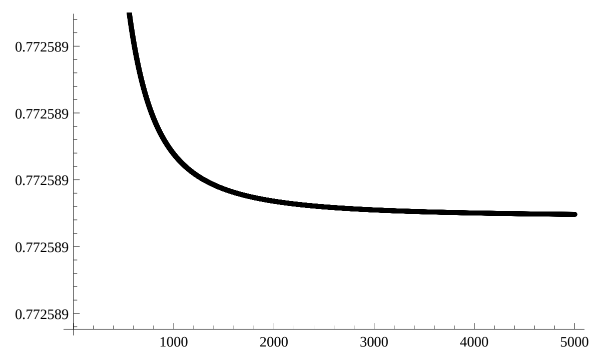

Figure 13: Expected number of additional 1.5–pairs for the solitaire Memory game when the number of pairs is increased by one, i.e., $e^{1.5}_{n,0} - e^{1.5}_{n-1,0}$. The value for $e^{1.5}_{5000,0} - e^{1.5}_{4999,0}$ is ≈ 0.77258872.

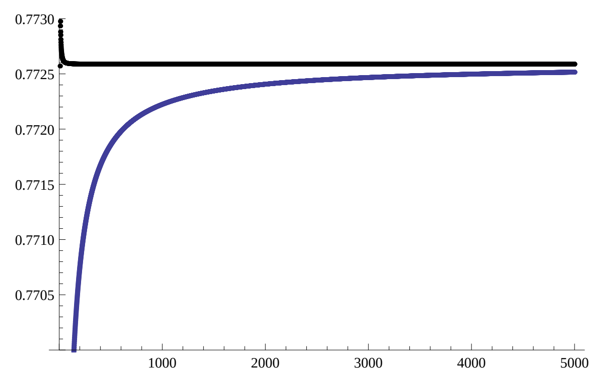

Figure 14: The graphs from Figure 12 and Figure 13 in one image. The upper black graph represents $e^{1.5}_{n,0} - e^{1.5}_{n-1,0}$, the lower blue one $e^{1.5}_{n,0}/n$.

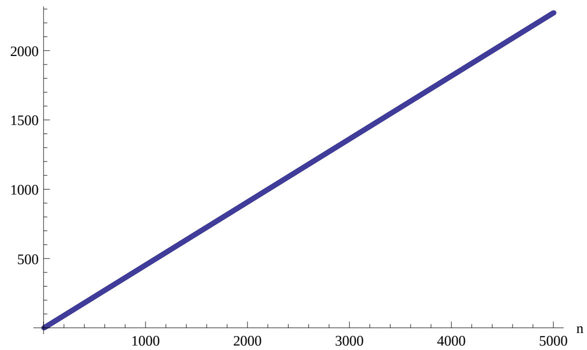

Figure 15: Expected number of 2.0–pairs multiplied by two for the solitaire Memory game using the optimal strategy from Lemma 4.1. The value for $n = 5000$ is ≈ 2273.4539.

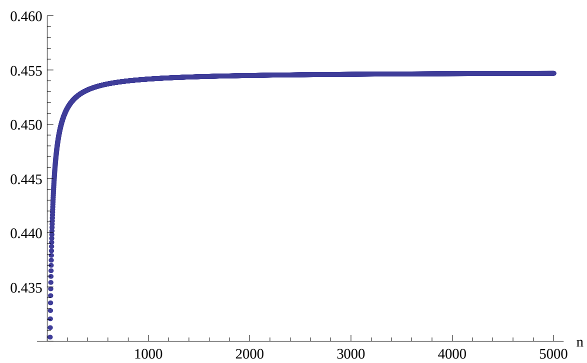

Figure 16: Expected number of 2.0–pairs multiplied by two for the solitaire Memory game using the optimal strategy from Lemma 4.1 – divided by n. The value for $n = 5000$ is ≈ 0.45469078.

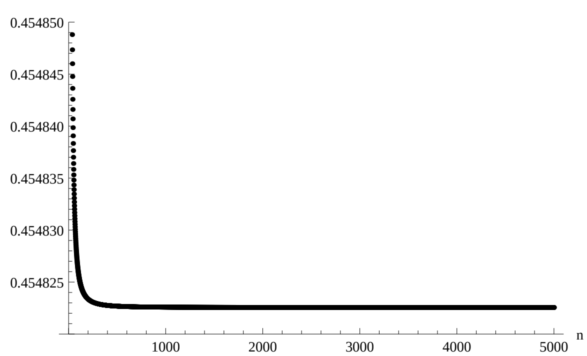

Figure 17: Expected number of additional 2.0–pairs multiplied by two for the solitaire Memory game when the number of pairs is increased by one, i.e., $e^{2.0}_{n,0} - e^{2.0}_{n-1,0}$. The value for $e^{2.0}_{5000,0} - e^{2.0}_{4999,0}$ is ≈ 0.45482256.

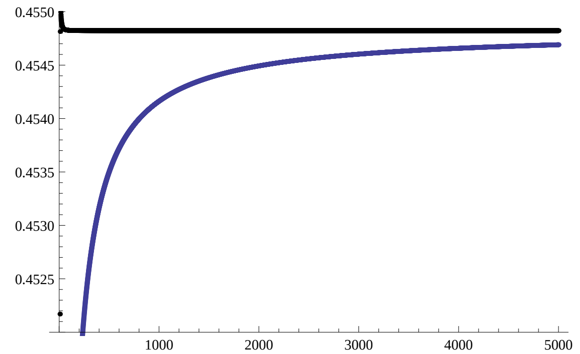

Figure 18: The graphs from Figure 16 and Figure 17 in one image. The upper black graph represents $e^{2.0}_{n,0} - e^{2.0}_{n-1,0}$, the lower blue one $e^{2.0}_{n,0}/n$.

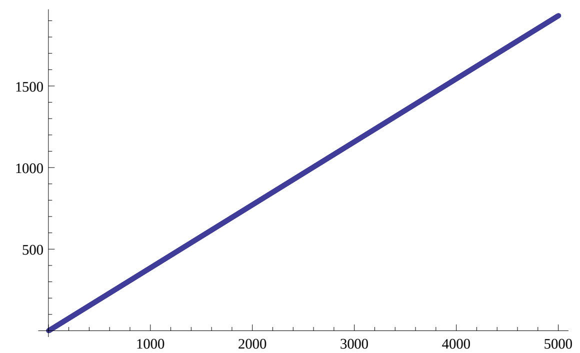

Figure 19: Expected number of other moves for the solitaire Memory game using the optimal strategy from Lemma 4.1. The value for $n = 5000$ is ≈ 1931.2899.

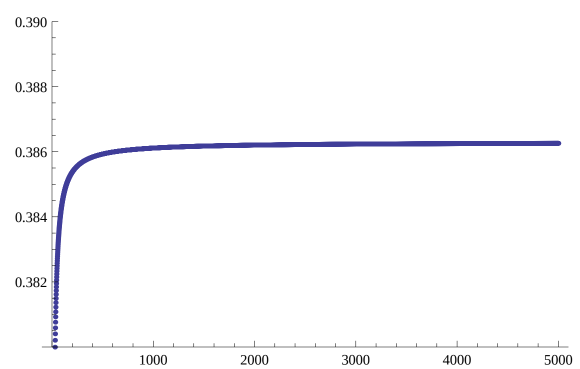

Figure 20: Expected number of other moves for the solitaire Memory game using the optimal strategy from Lemma 4.1 – divided by n. The value for $n = 5000$ is ≈ 0.38625799.

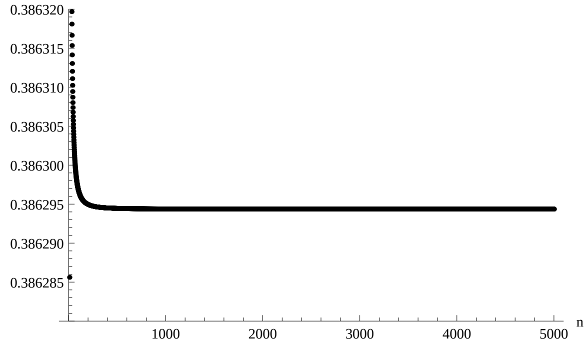

Figure 21: Expected number of other moves for the solitaire Memory game when the number of pairs is increased by one, i.e., $e^{other}_{n,0} - e^{other}_{n-1,0}$. The value for $e^{other}_{5000,0} - e^{other}_{4999,0}$ is ≈ 0.38629436.

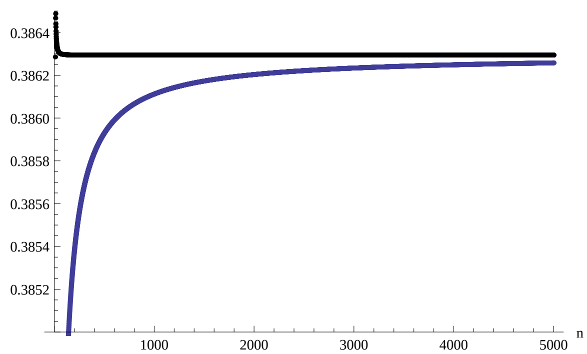

Figure 22: The graphs from Figure 20 and Figure 21 in one image. The upper black graph represents $e^{other}_{n,0} - e^{other}_{n-1,0}$, the lower blue one $e^{other}_{n,0}/n$.

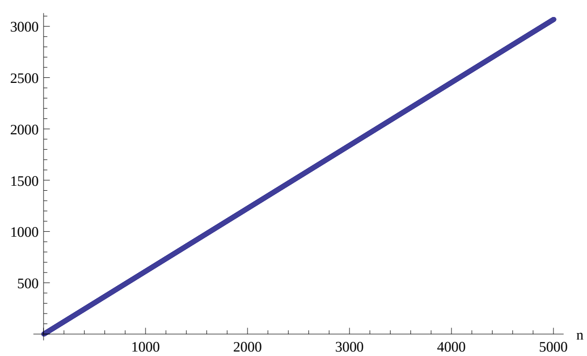

Figure 23: Expected number of moves where no pair is collected for the solitaire Memory game using the optimal strategy from Lemma 4.1. The value for $n = 5000$ is ≈ 3068.0169.

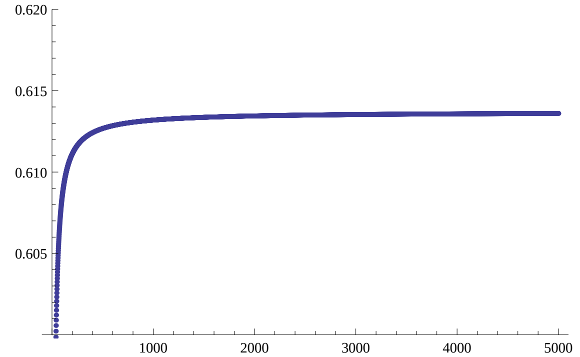

Figure 24: Expected number of moves where no pair is collected for the solitaire Memory game using the optimal strategy from Lemma 4.1 – divided by n. The value for $n = 5000$ is ≈ 0.61360338.

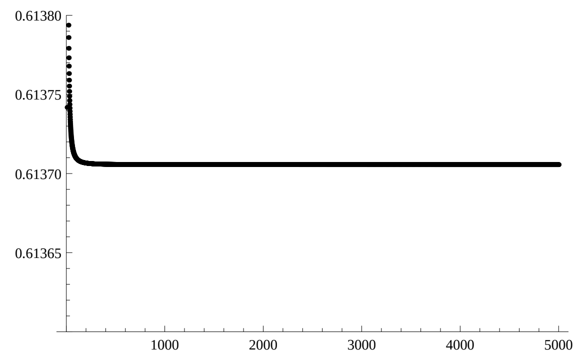

Figure 25: Expected number of moves where no pair is collected for the solitaire Memory game when the number of pairs is increased by one, i.e., $(\frac{1}{2}e^{2.0}_{n,0} + e^{other}_{n,0}) - (\frac{1}{2}e^{2.0}_{n,0} + e^{other}_{n-1,0})$. The value for $(\frac{1}{2}e^{2.0}_{5000,0} + e^{other}_{5000,0}) - (\frac{1}{2}e^{2.0}_{4999,0} + e^{other}_{4999,0})$ is ≈ 0.61370564.

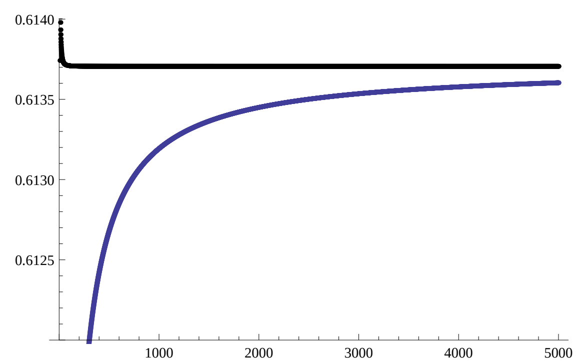

Figure 26: The graphs from Figure 24 and Figure 25 in one image. The upper black graph represents $(\frac{1}{2}e^{2.0}_{n,0} + e^{other}_{n,0}) - (\frac{1}{2}e^{2.0}_{n,0} + e^{other}_{n-1,0})$, the lower blue one $(\frac{1}{2}e^{2.0}_{n,0} + e^{other}_{n,0})/n$.